Problem Set 6

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 6 covers materials from §4.6 – §4.8.

1. Let f be a polynomial given by
$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n.$$
If the coefficients of f satisfy that
$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1} = 0,$$
show that f has a root in the interval $(0, 1)$.

2. Prove the following statements using Mean Value Theorem.
   (a) For every real numbers a and b, we have $|\cos a - \cos b| \leq |a - b|$.
   (b) For every $a, b \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have
$$|\tan a - \tan b| \geq |a - b|.$$
   (c) For every $t > -1$, we have
$$\ln(1+t) \geq \frac{t}{1+t}.$$

3. Suppose that a runner completes a 10 km race in 30 minutes. Assuming that the runner's speed at the finishing line is zero, explain why he must have been running at a speed of exactly 19 $\mathrm{km/h}$ at least twice during the race.

4. Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function and let $a < b$ be real numbers. By considering the function $g: \mathbb{R} \to \mathbb{R}$ defined by
$$g(x) = f(x) - f(b) - f'(x)(x-b) - \frac{f(a) - f(b) - f'(a)(a-b)}{(a-b)^2}(x-b)^2,$$
show that there exists $c \in (a, b)$ such that
$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2.$$

5. The following table shows some values taken by two functions $f, g: \mathbb{R} \to \mathbb{R}$, which are both differentiable everywhere on $\mathbb{R}$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	-1	1	-2	2
2	0	-1	0	0
5	2	3	-1	-2

For each of the following statements, determine **with explanation** whether it is true or false.
   (a) f does not have an inverse.
   (b) g' takes the value 1 at some number in the interval $[0, 2]$.

6. Find the coordinates of all the local extremum points of each of the following functions.

(a) $f(x) = x^2(x+4)^{\frac{2}{3}}$ (b) $f(x) = \dfrac{x^3+9x}{x^2+1}$ (c) $f(x) = \dfrac{|x|}{(x+1)^2}$

7. Sketch the graph of a function f which satisfies all the following conditions:

(i) The domain of f is $[0, +\infty)$.

(ii) f is continuous on $[0, 3)$ and on $[3, +\infty)$.

(iii) $\displaystyle\lim_{x\to 0^+} \frac{f(x) - 2}{x} = 1$.

(iv) $f'(x) > 0$ for every $x \in (0, 2) \cup (3, 4) \cup (4, +\infty)$.

(v) $\displaystyle\lim_{x\to 3^-} f(x) = -\infty$.

(vi) $\displaystyle\lim_{x\to 4} \frac{f(x) - 1}{x - 4} = 0$.

(vii) $f'(x) < 0$ for every $x \in (2, 3)$.

(viii) $f''(x) < 0$ for every $x \in (0, 3) \cup (3, 4)$.

(ix) $f''(x) > 0$ for every $x \in (4, +\infty)$.

(x) $\displaystyle\lim_{x\to +\infty} (f(x) - x) = -5$.

8. Sketch the graph of each of the following functions. As a guideline to your presentation, you may adopt the seven-step procedure described in Remark 4.60 in the lecture note.

(a) $f(x) = xe^{\frac{1}{x}}$

(b) $f(x) = \dfrac{(x+15)(x+1)^2}{(x-6)^2}$

(c) $f(x) = \left|\dfrac{x^2(x-3)}{x+1}\right|$

(d) $f(x) = |x-2|\sqrt{\dfrac{x}{x-1}}$

9. Let $f(x) = x^2 - \frac{8}{x-1}$ for every $x \neq 1$.

(a) Sketch the graph of f. You don't need to find the exact value of its x-intercept.

(b) Let $g(x) = f(|x|)$ for $|x| \neq 1$.

 (i) Explain whether g is differentiable at 0 or not.

 (ii) Sketch the graph of g.

10. Consider the functions $g: \mathbb{R} \setminus \{-1/2\} \to \mathbb{R}$ and $h: (-1, 1) \to \mathbb{R}$ defined by

$$g(x) = \frac{2x(1-x)}{2x+1} \qquad \text{and} \qquad h(x) = \arcsin\frac{2x}{1+x^2}.$$

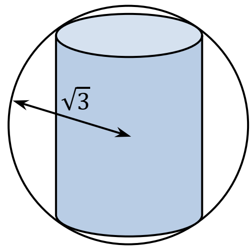
(a) Show that $g(0) = h(0)$ and $g'(0) = h'(0)$.

(b) Let f be the function defined by

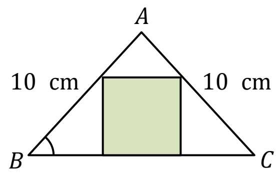
$$f(x) = \begin{cases} g(x) & \text{if } x < 0 \text{ and } x \neq -1/2 \\ h(x) & \text{if } 0 \leq x < 1 \end{cases}.$$

Sketch the graph of f.

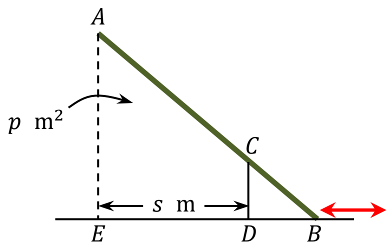
11. Find the height and the base radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$ units.



12. ΔABC is an isosceles triangle in which the lengths of the edges AB and AC are both 10 cm. Find the optimal size of $\angle ABC$ such that the square inscribed in ΔABC has the largest area.



13. The diagram below shows a straight rod AB of length 8 m resting on a vertical wall CD of height 1 m. The end B is free to slide back and forth along a horizontal rail such that AB is vertically above the rail and hangs on the wall. Let E be the vertical projection of A onto the rail. Denote the length of DE as s m and the area of the trapezium $CAED$ as p m^2.



(a) Find the maximum value of s.

(b) Explain whether p attains maximum when s attains its maximum.

14. When we solve optimization problems in class, we have often made use of the fact that if a function has **only one critical number** which is a local maximum, then this local maximum must also be a global maximum. Try to prove this fact:

⊙ Let $f: (a, b) \to \mathbb{R}$ be a differentiable function and suppose that f has its **only critical number** at $c \in (a, b)$. If f attains **local** maximum at c, show that f also attains **global** maximum at c.

Hint: Prove by contradiction. If f attains local maximum at c but does not attain global maximum at c, try to show using Rolle's Theorem that there must be another critical number of f apart from c.